1 *Good Morning PCMI*

Hey, welcome to the class. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Have fun!** Make sure you're spending time working on problems that interest you. Feel free to skip problems that you're already sure about. Relax and enjoy!
- **Teach only if you have to.** You may feel the temptation to teach others at your table. Fight it! We don't mean you should ignore your tablemates (please!!) but try not to deny anyone the chance to discover some Cool Math Stuff. You'll see what we mean.
- **Each day has its Stuff.** There are problem categories: Sketch of the Day, Important Stuff, Neat Stuff, and Tough Stuff. We'll have you do Sketch of the Day first thing, but after that you should check out Important Stuff first. We try to make sure that the problems in Important Stuff can be picked up regardless of how much or little work you've done on prior sets.

One question from a previous year turned out to be the unsolved Goldbach Conjecture. Nobody got that one.

On Day 3, go back and read these again.

Will you remember? Maybe!

Sketch of the Day

If you don't have a computer with you, now is a good time to make friends with those around you. Please share the computers! We encourage teamwork at PCMI.

PROBLEM

Get Sketchpad on your computer. The actual SotD will be done after the break. Get Sketchpad on your computer!

The word of the day is "logistical nightmare". Okay, I guess that's two words.

Important Stuff.

1. Factor.
(a) $x^2 - 12x + 35$
(b) $36m^2 - 72m + 35$

2. Find all solutions to each equation. Which equation is easiest to solve?
(a) $(x + 3)(x - 5) = -15$
(b) $(x + 3)(x - 5) = 0$
(c) $(x + 3)(x - 5) = 84$

Don't sweat it if you're not feeling too familiar with factoring problems. We'll get you there. Take some guesses, try stuff out, ask a tablemate for some pointers.

3. Find all solutions to each equation. Yes, you have to do all six parts.
(a) $x^2 - 12x + 32 = 0$
(b) $x^2 - 12x + 33 = 0$
(c) $x^2 - 12x + 34 = 0$
(d) $x^2 - 12x + 35 = 0$
(e) $x^2 - 12x + 36 = 0$
(f) $x^2 - 12x + 37 = 0$

4. Get Sketchpad on your computer!

5. A rectangle has perimeter 24 and area 33. What are its side lengths?

6. A rectangle has perimeter 24. What is its maximum possible area?

7. A rectangle has perimeter $(4x - 8)$. What is its maximum possible area?

Neat Stuff.

8. It's division time! Without a calculator, find the remainder when 60953 is divided by 145.

The long way? Yes.

9. It's division time! Without a calculator, find the remainder when $6x^4 + 9x^2 + 5x + 3$ is divided by $x^2 + 4x + 5$.

10. Find the remainder when each of these polynomials is divided by x^2.
(a) $5x^2 + 7x + 9$
(b) $3x^3 + 5x^2 + 7x + 9$
(c) $x^4 + 3x^3 + 5x^2 + 7x + 9$

11. Find the remainder when each of the polynomials from problem 10 is divided by $x^2 + 1$.

12. John says he can just look at a polynomial and tell you what its remainder will be when you divide by $x^2 + 1$. He might be doing some arithmetic too, but absolutely no division. Play around with a bunch of polynomials and see if you can figure out what he's doing.

John once factored a polynomial just for snorin' too loud. Hey, it's the Old West.

13. Let $p(x) = 3x^3 + 4x^2 + 5x + 7$ and $q(x) = x^3 + 10x^2 + 6x + 3$. For each expression, find the *remainder* when the expression is divided by $x^2 + 1$.
(a) $p(x)$
(b) $q(x)$
(c) $p(x) + 17$
(d) $p(x) + q(x)$
(e) $p(x) - q(x)$
(f) $p(x) \cdot q(x)$

14. Which of these polynomials is a multiple of $x^2 + 1$?
(a) $3x^4 + 2x^2 + 1$
(b) $x^6 + 4x^4 + 6x^2 + 3x + 5$
(c) $x^5 + 2x^4 + 3x^3 + 6x^2 + 2x + 4$
(d) $x^5 + 2x^4 + 3x^3 + 6x^2 + 2x + 5$

15. Which of these numbers is divisible by 101?
(a) 30201
(b) 1040635
(c) 123624
(d) 123625

Why would we even ask this question here? Or this one?

16. Describe a method for testing whether a number is divisible by 9.

We do *not* mean "Divide it by 9 and see if there is no remainder."

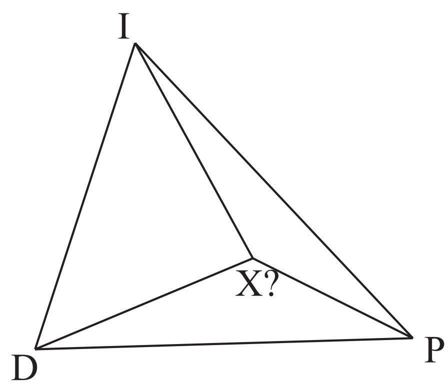
17. Describe a method for testing whether a number is divisible by 11.

18. Describe a method for testing whether a number is divisible by 99.

Sketch of the Day, Take 2

PROBLEM

Three cities get together to build an airport. They want to minimize the lengths of the new roads that need to be built.



Build this sketch, then use Sketchpad to figure out where the airport (point X) should be placed. What's so special about this point, anyway?

Tough Stuff.

19. (a) If $(x+3)(x-5) = N$ has exactly one solution, what is the value of N?

(b) If $(x-a)(x-b) = N$ has exactly one solution, find N in terms of a and b.

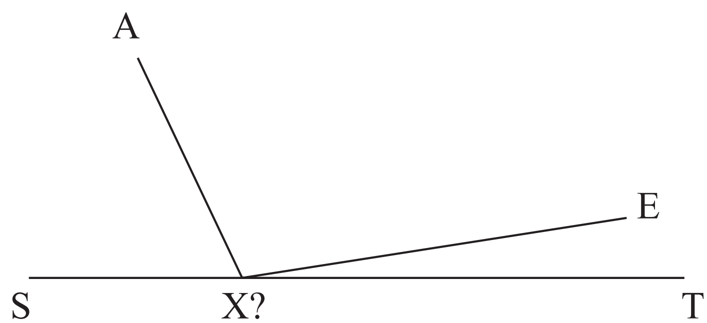
20. A triangle has perimeter 24. Find its maximum possible area, and explain how you know that this *must* be it.

21. Describe a method for testing whether a number is divisible by 101. (Yes, 101 is prime.)

2 *The PCMI Story*

Sketch of the Day

Art (at point A) sets his tent on fire, and wants to tell everyone (at point E) about his tragedy. However, he is extremely thirsty, and needs to run to nearby ST creek.



Where should he run toward (point X) to minimize the total distance $AX + XE$?

Work on this sketch right away. If you figure out where point X should be, start thinking about how you could *construct* point X more directly, or how you could *prove* that point X must be the right one.

Important Stuff.

1. Find all solutions to each quadratic equation.
(a) $w^2 - 22w + 121 = 0$
(b) $w^2 - 22w + 120 = 0$
(c) $w^2 - 22w + 119 = 0$
(d) $w^2 - 22w + 100 = 0$
(e) $w^2 - 22w + 2 = 0$
(f) $w^2 - 22w - 1 = 0$
(g) $w^2 - 22w - 408 = 0$
(h) $w^2 - 22w + 130 = 0$

Aw man, 8 of 'em? If only there were some sort of shortcut.

2. (a) A rectangle has perimeter 44 and area 100. What are its length and width?
(b) A rectangle has perimeter 44 and area 2. What are its length and width?

3. Two numbers add up to 64 and their product is 960. Given that $32^2 = 1024$, find the two numbers.

Why should we care that $32^2 = 1024$ here? And how does this make you feel?

4. Let $f(x) = x^3 + 3x^2 - 8x - 80$. Find the remainder when $f(x)$ is divided by each of these.
(a) $(x - 1)$
(b) $(x - 2)$
(c) $(x - 3)$
(d) $(x - 4)$
(e) $(x - 5)$

Feel free to uh, **divide** the work on this one among your group, there is no need to do all five of these yourself. But make sure you record all five answers.

5. Complete this table for $f(x) = x^3 + 3x^2 - 8x - 80$.

x	$f(x)$
0	
1	
2	
3	
4	
5	

6. Find the remainder when $f(x) = x^{12} + 3x - 1$ is divided by each of these.
(a) $(x - 1)$
(b) $(x - 2)$
(c) $(x - 10)$
(d) $(x + 1)$

7. Triangle BEV has points $B(2, 1)$, $E(4, 1)$, and $V(4, 6)$.
(a) Draw triangle BEV in the plane.
(b) New points are created from the points in triangle BEV according to the rule

$$(x, y) \mapsto (-y, x)$$

Draw the new triangle created in this way in the same plane, and describe how this triangle is related to the original.

8. Transform triangle BEV according to the rule

$$(x, y) \mapsto (-x, -y)$$

Draw the new triangle and describe how this triangle is related to the original.

Neat Stuff.

9. Jerry repeats the transformation in problem 7 a whole bunch of times.

(a) What happens after the transformation is applied twice?

(b) What happens after the transformation is applied three times?

(c) . . . four times?

(d) . . . five times?

(e) . . . thirteen times?

(f) . . . 102 times?

Poor Jerry. When your favorite number is 102, it can be tough.

10. (a) Sketch the graph of $y = x^2 - 10x + 29$.

(b) Explain why the graph of $y = x^2 - 10x + 29$ *cannot* cross the x-axis.

11. Josue thinks of two numbers with sum 8 and product 15. What is the sum of the squares of these numbers?

12. Cathy thinks of two numbers with sum 12 and product 32. What is the sum of the squares of these numbers?

13. The sum of two numbers is 12, and their product is 33. What is the sum of their squares?

We figured nobody would think of *these* numbers.

14. The sum of two numbers is 23, and their product is $14\frac{1}{2}$. What is the sum of their squares?

15. The sum of two numbers is s, and their product is p. What is the sum of their squares?

16. The sum of two numbers is 12, and their product is 73. What is the sum of their squares?

17. Wait a sec. The sum of the squares is what now? How is that even possible?

18. Roger takes triangle BEV and applies a wacky transformation:

$$(x, y) \mapsto (x + y, -3x + 7y)$$

(a) Draw this new triangle ROG. . . if it is a triangle? Is it even a triangle anymore?

(b) What is the area of this new shape? How does ROG compare (in area) to BEV?

19. Function $a(n)$ takes whole number inputs:

$$a(n) = (\sqrt{3})^n + (-\sqrt{3})^n$$

(a) Tabulate $a(n)$ for $n = 0$ through 6.
(b) Find $a(10)$ and $a(101)$.

A *whole number* is 0, 1, 2, 3, 4, etc. As opposed to a *hole number*, which is only 0.

20. The sum of two numbers is s and the product is p. Find the sum of the *cubes* of the two numbers, in terms of s and p.

Tough Stuff.

21. The quadratic equation $x^2 - 10x + 22 = 0$ has two roots.
(a) Find a quadratic whose roots are the *squares* of the roots of $x^2 - 10x + 22 = 0$.
(b) Find a quadratic whose roots are the nth powers of the roots of $x^2 - 10x + 22 = 0$.

22. Yesterday we found that there's a point inside most triangles that forms three 120° angles with segments to the three vertices. A *Matsuura triangle* is a triangle whose side lengths are all integers, *and* whose three interior segment lengths from the 120° point are also integers. Find some Matsuura triangles, or prove they do not exist.

23. Find all integer solutions to this system of equations:

$$\begin{aligned} a + b &= cd \\ c + d &= ab \end{aligned}$$

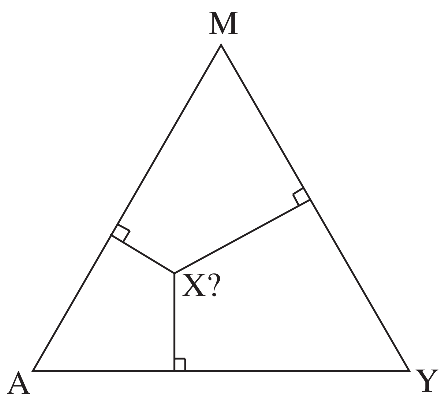
Apparently there was a call to make the Tough Stuff tougher; so, have fun!

3 *Moon Over PCMI*

As the title suggests, we worked on this problem set into the night! Don't do that yourself; enjoy your afternoon off!

Sketch of the Day

Equilateral triangle YAM has point X inside it.



Where should point X be placed so that the three *perpendicular* segments to the three sides of the triangle have a minimum total length?

Important Stuff.

1. (a) Two numbers add up to 30, and their product is 161. What are the two numbers?

(b) Two numbers add up to 30, and their product is 289. What are the two numbers?

Say, isn't 289 a perfect square? 8-15-17, or something like that? Oh, never mind. That's not a real problem, anyway.

2. Let $a(x) = -x^3 + 2x^2 + 3x + 5$ and $b(x) = 3x^3 - x^2 + 8x - 8$. Find the *remainder* when each of these expressions is divided by $x^2 + 1$.
(a) $a(x)$
(b) $b(x)$
(c) $a(x) + b(x)$
(d) $a(x) - b(x)$
(e) $3a(x) + 3b(x)$
(f) $a(x) \cdot b(x)$

3. Let $p(x) = x^4 - 2x^2 + 3$ and $q(x)$ is the *remainder* when $p(x)$ is divided by $(x-2)(x-5)$... in other words, $x^2 - 7x + 10$.
(a) Complete this table of values for $p(x)$:

x	$p(x)$
0	
1	
2	
3	
4	
5	

(b) Complete this table of values for $q(x)$:

x	$q(x)$
0	
1	
2	
3	
4	
5	

Notice anything interesting?

4. Let $f(x) = 3x^2 - 10x + 21$. Find a linear function that agrees with $f(x)$ when $x = 3$ and when $x = -5$.

5. Let $f(x) = 3x^2 - 10x + 21$. Ooh, the same function.
(a) Use division to find $g(x)$, a linear function that agrees with $f(x)$ when $x = 3$ and when $x = 3$.
(b) Graph $f(x)$ and $g(x)$ on the same axes, using Sketchpad or a graphing calculator. What do you notice?

Who in the what now? This isn't a typo, everyone.

6. Let $a = 2 + 3i$ and $b = 5 - 7i$ be complex numbers. Find each of these.
(a) $a + b$
(b) $a - b$
(c) $3a + 3b$
(d) ab (the product)

Do this like regular algebra, except whenever you see i^2, replace it by -1. That's the big property of i, it's the square root of -1.

7. (a) Go back and do the "BEV" problems (numbers 7 and 8) from Day 2 if you haven't already.

(b) Anita takes triangle BEV and performs the transformation

$$(x, y) \mapsto (3x - 2y, -5x + y)$$

Draw the new shape; is it even a triangle? What is the area of the new shape, and how does the area compare to the area of BEV?

Neat Stuff.

8. Look back at today's SotD, and do it again with a generic triangle TIM instead of the equilateral triangle used in the original sketch. What happens?

9. Find the equation of the tangent line to $f(x) = x^3$ at $x = 2$. Do *not* use calculus!

If this doesn't make much sense, go back and check out problem 5. It might give you some help.

10. Find the equation of a parabola that is tangent to the function $f(x) = x^4 - 2x^2 + 3$ at $x = 1$, and also intersects $f(x)$ at $x = -1$. Under no circumstances are you to use calculus for this problem! We'll know!

We were torn on this one, it's almost Tough Stuff. Note our insistence.

11. $B(n)$ takes whole number inputs according to the rule

Troy says that if 0 is a whole number, then 8 should be a two-whole number.

$$B(n) = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n$$

(a) Tabulate $B(n)$ for $n = 0, 1, 2, 3, 4, 5$. Anything surprising?

(b) Find an equation with integer coefficients that has $1 + \sqrt{2}$ as a root. (Say, what might another root need to be?)

(c) $B(n)$ can be tabulated according to a two-term recursive rule:

$$B(n) = c \cdot B(n-1) + d \cdot B(n-2)$$

Find c and d.

This term of $B(n)$ is c times the last one, plus d times the one before that.

12. $C(n)$ takes whole number inputs according to the rule

$$C(n) = (2 + i)^n + (2 - i)^n$$

(a) Tabulate $C(n)$ for $n = 0, 1, 2, 3, 4, 5$. Anything surprising?

(b) Find an equation with integer coefficients that has $2 + i$ as a root.

(c) $C(n)$ can be tabulated according to a two-term recursive rule:

$$C(n) = p \cdot C(n-1) + q \cdot C(n-2)$$

Find p and q.

13. The sum of two numbers is s and the product is p. Find the sum of the…

(a) squares of the two numbers.

(b) cubes of the two numbers.

(c) fourth powers of the two numbers.

(d) …a generalization?

Tough Stuff.

14. Let ABC be a triangle with $AB = 7$, $AC = 19$, $BC = \sqrt{313}$. Define function f on the plane by the rule

$$f(X) = AX + BX + CX$$

Find the minimum value of $f(X)$ and describe where it occurs.

15. On Day 1 we found that there's a point inside most triangles that forms three 120° angles with segments to the three vertices. A *Matsuura triangle* is a triangle whose side lengths are all integers, *and* whose three interior segment lengths from the 120° point are also integers. Find some Matsuura triangles, or prove they do not exist.

We'll keep these on the problem sets until someone gets 'em right!

16. Find all integer solutions to this system of equations:

$$\begin{aligned} a + b &= cd \\ c + d &= ab \end{aligned}$$

17. Given a positive integer k, there are a number of values of b so that the quadratic $x^2 + bx + kb$ is factorable (over the integers). Find a function that determines, based on k, how many such values of b there are.

4 *Sleepless in PCMI*

Sketch of the Day

Sketch a right triangle that stays a right triangle when you move its points around. (What would you need to construct first?)

Then, build three equilateral triangles on the outside of the right triangle by using each side of the right triangle as the base for an equilateral triangle.

Find a relationship between the areas of the equilateral triangles.

Writing problem sets is hard work, and making them even remotely funny is harder! Please, let us know if you get a joke! We'd really love to hear about it.

James: Please do knot use PVC to make these triangles. To everyone: Feel free to play around with this sketch a little if you like, there is more to find.

Important Stuff.

1. Let $w = 4 + i$ and $z = 6 - 5i$. Calculate each of these. Remember that $i^2 = -1$.
(a) $w + z$
(b) $w - z$
(c) $3w - 2z$
(d) w^2
(e) wz

2. Here are two functions:

$$w(x) = x^3 - x^2 + 2x + 3$$

$$z(x) = x^3 - 3x^2 - 4x + 3$$

Calculate the remainder when each of these expressions is divided by $x^2 + 1$.
(a) $w(x)$

"is"? Is what? I can't stand the suspense!

Feel free to use any shortcuts you've found previously, on Set 1 or 3.

(b) $z(x)$
(c) $w(x) + z(x)$
(d) $3w(x) - 2z(x)$
(e) $[w(x)]^2 = x^6 - 2x^5 + 5x^4 + 2x^3 - 2x^2 + 12x + 9$
(f) $w(x) \cdot z(x) = x^6 - 4x^5 + x^4 + 4x^3 - 20x^2 - 6x + 9$

3. Find the remainder when each of these is divided by x^2+1.
(a) x^2
(b) x^3
(c) x^4
(d) x^5
(e) x^6
(f) x^7
(g) x^8
(h) x^{13}
(i) x^{102}

4. Here are some transformation rules for points or objects in the plane. Describe what each rule does, as accurately as you can.
(a) $(x, y) \mapsto (y, x)$
(b) $(x, y) \mapsto (-x, y)$
(c) $(x, y) \mapsto (y, x)$ and then *that new* $(x, y) \mapsto (-x, y)$
(d) $(x, y) \mapsto (x - y, x + y)$
(e) $(x, y) \mapsto (4x - y, y + 4x)$

You might draw a figure with your name on it, such as triangle PEG. Poor Jo can only draw a line segment with her name on it.

Neat Stuff.

5. Let $f(x) = x^3 - 6x^2 + 4x + 8$. Find the remainder when $f(x)$ is divided by each of these.
(a) x^2
(b) $(x - 1)^2$
(c) $(x - 2)^2$
(d) $(x - 3)^2$

6. On a graphing calculator (or Sketchpad), use the *same axes* to graph $f(x) = x^3 - 6x^2 + 4x + 8$ and, one by one, the remainders from problem 5. What do you notice?

Sketchpad can do this too? Wahoo, it works!

7. Find the equation of the tangent line to $f(x) = x^4$ at $x = 1$.

8. A line segment is drawn from the point $(0, 7)$ to somewhere on the x-axis $(p, 0)$, then from there to the point $(13, 3)$.

(a) Draw this situation.

(b) Debbie wonders if the total distance of these segments can be written as a function of p. It would be

$$D(p) = \cdots$$

Use Mr. Pythagoras to figure it out.

(c) Use a graphing calculator to estimate or find the value of p that gives the minimum total distance.

We'd do it, but we're supremely lazy. So now you're stuck with it.

9. If you're interested, revisit problem 8 from the perspective of similar triangles, and find a general formula for the "best point" in terms of the nonzero values given in the problem.

10. The graph of $y = 1 - x$ is a line. Describe what happens when this line is transformed by each of these rules.

(a) $(x, y) \mapsto (x + 3, y)$

(b) $(x, y) \mapsto (x, y + 3)$

(c) $(x, y) \mapsto (x, 3y)$

(d) $(x, y) \mapsto (-y, x)$

(e) $(x, y) \mapsto (x^2, y^2)$

(f) $(x, y) \mapsto (\sqrt{x}, \sqrt{y})$

11. In the SotD from Day 2, Art runs just as fast to the creek as he does away from it. But in reality, Art runs twice as fast when he's thirsty as he does when he's not. How does this change the sketch and its solution? Can you model this in Sketchpad? on a graphing calculator? by making Art thirsty? On a related note, what is Snell's Law?

12. $B(n)$ takes whole number inputs according to the rule

$$B(n) = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n$$

(a) Tabulate $B(n)$ for $n = 0, 1, 2, 3, 4, 5$. Anything surprising?

(b) Find an equation with integer coefficients that has $1 + \sqrt{2}$ as a root. (Say, what might another root need to be?)

(c) $B(n)$ can be tabulated according to a two-term recursive rule:

$$B(n) = c \cdot B(n - 1) + d \cdot B(n - 2)$$

Find c and d.

13. $C(n)$ takes whole number inputs according to the rule

$$C(n) = (2 + i)^n + (2 - i)^n$$

(a) Tabulate $C(n)$ for $n = 0, 1, 2, 3, 4, 5$. Anything surprising?
(b) Find an equation with integer coefficients that has $2 + i$ as a root.
(c) $C(n)$ can be tabulated according to a two-term recursive rule:

$$C(n) = p \cdot C(n-1) + q \cdot C(n-2)$$

Find p and q.

14. The sum of two numbers is s and the product is p. Find the sum of the...
(a) squares of the two numbers.
(b) cubes of the two numbers.
(c) fourth powers of the two numbers.
(d) ...a generalization?

Tough Stuff.

15. So we've discovered that this 120° point gives the best possible total distance to the three vertices. But what about other points? They're worse, but some are not much worse. Indeed, the shape of the points that are *equally bad* is interesting. What's it look like? What's it look like if you move outside the original triangle?

16. Find several triangles that have integer side lengths (with no common factors) and a 120° angle. Generalize?

17. Find some Matsuura triangles, or prove they do not exist. (See previous sets for the definition.)

18. Given a positive integer k, there are a number of values of b so that the quadratic $x^2 + bx + kb$ is factorable (over the integers). Find a function that determines, based on k, how many such values of b there are.

19. Find this sum exactly:

$$0 + \frac{1}{100} + \frac{4}{10000} + \frac{9}{1000000} + \cdots + \frac{n^2}{10^{2n}} + \cdots$$

5 *Deep in the Heart of PCMI*

Sketch of the Day

Redo the construction of Day 4 using a scalene triangle as the starter, building equilateral triangles along each side.

This sketch can be used to *construct* the 120° point! See if you can figure out how. If you've already built the construction one way, try to find another. You might try using ideas from Day 3's sketch!

All together now: The stars at night, are big and bright...

Important Stuff.

1. Find the remainder when each of these is divided by x^2+1.
(a) $A(x) = x^3 + 7x^2 - 11x + 12$
(b) $B(x) = x^4 + 7x^3 + 13x^2 + 7x + 12$
(c) $C(x) = A(x) + B(x)$
(d) $D(x) = A(x) \cdot B(x)$

What, you want us to multiply it out for you again? Or do we really need to do that?

2. Simplify each of these seemingly nasty-looking expressions that involve square roots of square roots of negative numbers. (Ew?)
(a) $\sqrt{(3+4i)(3-4i)}$
(b) $\sqrt{(5-12i)(5+12i)}$
(c) $\sqrt{(15+8i)(15-8i)}$
(d) $\sqrt{(x+yi)(x-yi)}$

3. How far is each of these points from the origin?
(a) $(3, 4)$
(b) $(5, -12)$

(c) $(15, 8)$
(d) (x, y)

4. (a) Draw a graph of all the points (x, y) that are 5 units from the origin.
(b) Write an equation for the graph you just drew.

5. Find all 12 complex numbers $a + bi$ with integers a, b so that

$$\sqrt{(a+bi)(a-bi)} = 5$$

A complex number with integer a, b is called a *Gaussian integer*. Knowing is half the battle. Oh, and we already gave you one of the answers.

Sergio watches Peter walk around a circle. The circle's radius is 1 meter. Sergio stands at the center, and Peter begins walking counter-clockwise. Consider a coordinate grid, with sergiO standing at the origin $O(0, 0)$ and Peter starting at the point $P(1, 0)$.

As Peter walks, Sergio watches Peter and keeps track of the angle he's turned.

Sketchpad anyone?

6. (a) Explain why the distance between Sergio and Peter is always 1.
(b) If Peter is at coordinates (x, y), write an equation to express the fact that Peter is 1 unit away from $(0, 0)$.

7. Describe, as completely as possible, how the y-coordinate of Peter's location changes as Sergio's angle increases.

8. Complete this table, giving the coordinates of Peter's location when Sergio has turned each angle.

Psst: Square root of 2 over 2!

Angle	Coordinates
$0°$	$(1, 0)$
$45°$	
$90°$	
$135°$	
$180°$	$(-1, 0)$
$225°$	
$270°$	
$315°$	
$360°$	
$405°$	
$450°$	

Neat Stuff.

9. Square $IRMA$ has one vertex at $A(1, 0)$. The center of the square is the origin $O(0, 0)$. Find the coordinates of the other vertices.

10. Find all four solutions to the equation $x^4 - 1 = 0$.

11. Equilateral triangle BND has one vertex at $B(1, 0)$. The center of the triangle is the origin $O(0, 0)$. Find the coordinates of the other vertices.

As Jerry mentioned yesterday, all four major "center" points of an equilateral triangle are the same spot.

12. Find all three solutions to the equation $x^3 - 1 = 0$.

13. Regular hexagon $DUGLAS$ has one vertex at $A(1, 0)$. The center of the hexagon is the origin $O(0, 0)$. Find the coordinates of the other vertices.

Nice job with that knotty problem A1, Douglas!

14. Find all six solutions to the equation $x^6 - 1 = 0$.

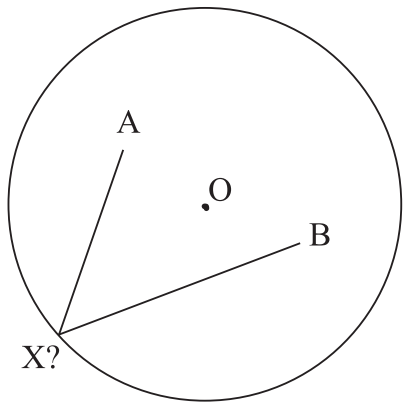
15. Let $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. Take powers of z until you notice something.

16. Find a complex number z so that $z^{12} = 1$ but no smaller positive integer n has $z^n = 1$. Oh, why not just find them all instead.

It's Friday Bonus Sketches!

17. A magic voice from Cincinnati said that if you draw the altitudes from any interior point to these other figures' sides, they add up to a constant. (Just like Day 3's equilateral triangle.) Can you prove 'em? Oh, we also threw in some more that might not even be true, so you'll have to figure out which ones are for real.
(a) a square
(b) a rectangle
(c) a rhombus
(d) a parallelogram
(e) a regular pentagon
(f) an equilateral hexagon (not regular)
(g) an equiangular hexagon (not regular)

18. Points A and B are inside a circle. Where should point X be placed *on* the circle so that $AX + XB$ is as small as possible?



Tough Stuff.

19. Complete this long division problem, where all the missing digits are marked with an X. (There is no remainder.)

```
             x x 8 x x
        ------------------
  x x x | x x x x x x x x
            x x x
          -------
            x x x x
              x x x
            -------
                x x x x
                x x x x
                -------
```

20. Find several triangles that have integer side lengths (with no common factors) and a 120° angle. Generalize?

21. Find some Matsuura triangles, or prove they do not exist. (See previous sets for the definition.)

22. Suppose n is a positive integer. Is there a right triangle with rational numbers as its side lengths with area n?

6 *WKRP in PCMI*

Hey, welcome back to the class. We know you'll continue to learn a lot of mathematics here—some new tricks, some new perspectives on things you might already know about. A few things to recall about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly. Note: this is not a personal challenge! Please spend time considering looking at groups of questions for the connections and meaning of things, even between different sets or topics.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences). There is no such thing as going "too slow" on the problem sets.
- **Have fun!** Make sure you're spending time working on problems that interest you. Feel free to skip problems that you're already sure about. Relax and enjoy!
- **Teach only if you have to.** You may feel the temptation to teach others at your table. Fight it! It is far more useful to discuss strategies or observations, even on a question you might think you already know. We think you'll be surprised by what everyone in your group can come up with.
- **Each day has its Stuff.** We try to make sure that the problems in Important Stuff can be picked up regardless of how much or little work you've done on prior sets.

On Day 8, go back and read these again.

One question from a previous year turned out to be the Reimann Hypothesis. Nobody got that one. Look for it on Day 11.

At around midnight last night we decided that 510,510 is a cool number. We hope you agree.

Will you remember? Probably not!

Sketch of the Day

A *golden rectangle* is a rectangle whose length is exactly $\frac{1+\sqrt{5}}{2}$ times its width. (Among other properties.)

Construct a golden rectangle that stays a golden rectangle when you move its vertices around. You'll have to do something about that $\sqrt{5}$...

Important Stuff.

1. The *conjugate* of the complex number $z = a + bi$ is $\overline{z} = a - bi$.

(a) If $z = 5 + 2i$, what is $\overline{z}$? What is $\overline{\overline{z}}$, the conjugate of the conjugate?

(b) Let $w = 3 - 4i$. Calculate $w + \overline{w}$ and $w\overline{w}$.

(c) Find a complex number v so that $v + \overline{v} = 14$.

(d) Find a complex number v so that $v\overline{v} = 65$.

(e) Find a complex number v so that $v + \overline{v} = 14$ *and* $v\overline{v} = 65$.

Just let it, it's okay. By the way, $w\overline{w}$ is just w multiplied by $\overline{w}$.

2. Find two numbers whose sum is 14 and whose product is 65.

We're told you can solve this problem by "looking it up".

3. The *magnitude* of the complex number $z = a + bi$ is $|z| = \sqrt{z\overline{z}}$.

(a) Find the magnitude of $z = 5 + 2i$ and of $w = 3 - 4i$.

(b) Rewrite the equation $|z| = \sqrt{(a + bi)(a - bi)}$ as something that doesn't have i in it.

(c) Can the magnitude of a complex number ever be zero? negative?

(d) Find a complex number whose magnitude is $\sqrt{65}$.

And don't just change the i into some other letter like j, that doesn't cut it.

4. For each point, find its distance from the origin $(0, 0)$.

(a) $R(5, 2)$

(b) $U(3, -4)$

(c) $D(8, 1)$

(d) $Y(a, b)$

5. Let $z = 3+2i$. Find the *magnitude* of each of these complex numbers.

(a) z

(b) z^2

(c) z^3

(d) z^4

Why can't z just pick a value and stick to it? It's almost as bad as that x that keeps changing.

Neat Stuff.

6. Consider the transformation rule

$$(x, y) \mapsto (3x - 2y, 2x + 3y)$$

For each point, calculate the new point created by this transformation rule.
(a) $R(1, 0)$
(b) $N(3, 2)$
(c) $E(5, 12)$
(d) $L(-9, 46)$

7. Consider the numbers $v = 4 + 6i, b = 2 + i, e = 4 + i$.
(a) Calculate the results when each of v, b, and e is multiplied by i.
(b) Calculate the results when each of v, b, and e is multiplied by i *twice*.
(c) Calculate the results when each of these numbers is multiplied by i three times, four times, five times, thirteen times, 102 times.

This "e" is not the 2.718... number, just a letter used to represent the number $4 + i$.

Multiplying by i twice is *not* the same as multiplying by $2i$!

8. The notation $|z|$ that is used for magnitude is the same symbol as the notation used for absolute value. Does magnitude work in the same way absolute value works? Put another way, will real numbers have the same magnitude as their absolute value?

Real numbers *are* complex numbers, they're just numbers like $7 + 0i$ or $-\sqrt{2} + 0i$.

9. Find a complex number $w = a + bi$ with a, b positive integers and $|w| = 65$.

10. How many complex numbers $a + bi$ have integer a, b and magnitude $\sqrt{65}$?

11. A *primitive Pythagorean triple* is a set of three positive integers a, b, c with $a^2 + b^2 = c^2$ where a, b, c share no common factors besides the blatantly obvious.
(a) Find a primitive triple with hypotenuse 65.
(b) Find *both* primitive triples with hypotenuse 85.

6, 8, 10 is *not* one of these, nor is 60, 80, 100.

Last week you learned that dividing by $x^2 + 1$ and looking at the remainders is very similar to working with the complex number i. This next set of problems looks at remainders when dividing by $x^2 + x + 1$, a slightly but significantly different polynomial.

Polynomials are not to be confused with the rare disease *polynomila*, typically caused by misspelling mathematical words too often.

12. If $A(x) = x^3 + x^2 + 2x - 3$ and $B(x) = x^3 - x + 1$, find the remainder of each expression when dividing by $x^2 + x + 1$.
(a) $A(x)$
(b) $B(x)$
(c) $A(x) + B(x)$
(d) $A(x) \cdot B(x) = x^6 + x^5 + x^4 - 3x^3 - x^2 + 5x - 3$

13. Find the remainder when each of these is divided by $x^2 + x + 1$.
(a) x^2
(b) x^3
(c) x^4
(d) x^5
(e) x^6
(f) x^{13}
(g) x^{102}

For Algebra 2 teachers: Have you seen the expression $x^2 + x + 1$ hanging around anywhere before? Could this help explain what is going on? What do these weird questions even mean, anyway?

14. What connection could there be between $x^2 + x + 1$ and equilateral triangles?

Tough Stuff.

15. Find a number n that is the hypotenuse of exactly *four* primitive Pythagorean triples.

16. Eight primitive Pythagorean triples?!

17. Describe what types of positive integers n can be written in the form $n = a^2 + b^2$ for integer a, b. For example, 450 can be written this way: $450 = 21^2 + 3^2$.

18. The number 450 has eight odd factors: 1, 3, 5, 9, 15, 25, 45, 75, 225. Odd factors can be split into factors in the form $4k + 1$ and $4k + 3$. 450 has 6 odd $4k + 1$ factors and 2 odd $4k + 3$ factors.
Well, it turns out there's a remarkable connection between the number of these types of factors and the *number of different ways* 450 (or any positive integer) can be written in the form $n = a^2 + b^2$ for integer a, b. 450 can be written many different ways ($a = 3, b = 21$ is one of them).
So, your task is to figure out what the rule is—then prove that it works (harder). Don't forget that a or b can be negative or zero.

19. Matsuura triangles, anyone?

FREE SLURPEE DAY! FREE SLURPEE DAY!

7 *Fear and Loathing in PCMI*

Sketch of the Day

Here are the vertices of a parallelogram: $O(0,0), N(5,3), I(1,4), C(6,7)$. Find the area of the parallelogram by any means necessary. This is a good topic for table discussions to see what people did, and what other ways might be possible.

Generalize to find the area of a parallelogram with vertices $O(0,0)$, $L(a,b), E(c,d), D(a+c, b+d)$ in terms of a, b, c, and d.

Aki says hi.

More to look for: what point P inside the parallelogram minimizes the total of the *four* distances to the vertices?

Important Stuff.

1. Consider the complex numbers $e = 4+i, b = 2+i, v = 4+6i$.
 (a) Plot and label e, b, and v in a complex plane.
 (b) Multiply each number by i, then plot and label each of the new numbers in the *same* complex plane.
 (c) Multiply each of e, b, and v by i *twice*, then plot and label each of the new numbers in the same plane.
 (d) Three times? Four times? Five times? Thirteen times? 102 times?

You mgiht be experiencing a little deja vu here. Good!

If only Jerry's favorite number had been 3 instead of 102, everything would have been so much easier. . .

2. Let $z = 1+i$. Plot each of these in the same complex plane, and find the magnitude of each.
 (a) z
 (b) z^2
 (c) z^3
 (d) z^4

(e) z^5

3. Anastasia stands at the origin $(0,0)$ and stares at the powers of $1+i$ as they are built. Describe what happens to the powers from her perspective: where do they go? how far away (magnitude, anyone)?

4. (a) Find the magnitude of $5+2i$.
(b) Find the magnitude of $(5+2i)^2$.
(c) Find a Pythagorean triple with hypotenuse 29.

5. Consider the complex number $z = \frac{3}{5} + \frac{4}{5}i$. Plot and label each of these on the same complex plane.
(a) z
(b) z^2
(c) z^3
(d) z^4
(e) z^5

6. Connie stands at the origin $(0,0)$ and stares at the powers of $\frac{3}{5} + \frac{4}{5}i$ as they are built. Describe what happens as accurately as you can. How does it compare to what happens with the powers of $1+i$? Why is it different?

7. If $a = 4+3i$ and $b = 5+12i$, find the magnitude of
(a) a
(b) b
(c) $2a$
(d) $-b$
(e) ab

Neat Stuff.

8. (a) Find the magnitude of $w = 2+i$.
(b) Perform an operation to w that results in another complex number that has magnitude 5.

YES WE MEAN 5.

9. Do math to take each of these complex numbers and produce a primitive Pythagorean triple.

Triples are fun, whee! Don't get lost on this problem.

(a) $4+i$
(b) $8+3i$
(c) $15+4i$
(d) $16+7i$
(e) $23+2i$
(f) $42+9i$

10. Consider the transformation rule

$$(x, y) \mapsto (\frac{3}{5}x - \frac{4}{5}y, \frac{4}{5}x + \frac{3}{5}y)$$

Starting with the Peter point $P(1, 0)$, what happens as you apply this transformation repeatedly? How best could this transformation be described?

11. So you know the magnitude of $3 + 4i$ is 5, right? Can you find *all* the Pythagorean triples with hypotenuse 125?

12. Use the results of problem 7 to find a Pythagorean triple with hypotenuse 1105. Then another one!

What's so special about 1105 anyway?

13. Let a be a complex number with magnitude 5, and b be a complex number with magnitude 13. Consider $a + b$: what could its magnitude be? Could it be 10? 20? 0? 13? Is there a theorem at play here?

14. If $A(x) = x^3 + x^2 + 2x - 3$ and $B(x) = x^3 - x + 1$, find the remainder of each expression when dividing by $x^2 - x + 1$.

(a) $A(x)$
(b) $B(x)$
(c) $A(x) + B(x)$
(d) $A(x) \cdot B(x) = x^6 + x^5 + x^4 - 3x^3 - x^2 + 5x - 3$

If you didn't do the related problems on Day 6, go back and do those before trying these. We hope they're interesting for ya.

15. Find the remainder when each of these is divided by $x^2 - x + 1$.

(a) x^2
(b) x^3
(c) x^4
(d) x^5
(e) x^6
(f) x^7
(g) x^{13}
(h) x^{102}

For Algebra 2 teachers: Have you seen the expression $x^2 - x + 1$ hanging around anywhere before? Could this help explain what is going on? Are you even reading this?

16. What connection could there be between $x^2 - x + 1$ and a particular regular polygon?

Topological Stuff.

17. For each of these objects, count the number of **vertices**, **edges**, and **faces** of the objects.

(a) a cube

(b) a tetrahedron
(c) a cylinder
(d) a square pyramid
(e) a dodecahedron
(f) a football
(g) a shape provided by Joyce (bonus points available by degree of difficulty)

Not that football, PCMI is an international conference. Go Portugal or France, or something.

18. Make a table with the number of vertices, edges, and faces for these objects, or any others you enjoy. (A sphere, perhaps?) Is there a meaningful relationship between these numbers that holds no matter what solid you start with?

19. What about a graph in the plane, like Day 1's sketch, or even just a parallelogram? Is there some pattern to the vertices, edges, and faces there?

Here a **face** would just be an enclosed area in the plane.

Tough Stuff.

20. $\frac{J}{RE} + \frac{O}{GI} + \frac{Y}{NA} = 1$. Here, each letter is a unique and distinct number from 1 to 9. Find a solution!

Wait, are you reading these problems before doing and grokking the rest of them? Get back to the first page, NOW!

21. As you keep taking powers of $z = \frac{3}{5} + \frac{4}{5}i$, will they eventually wrap around onto themselves? In other words, are there powers k and m with $z^k = z^m$ for this z?

22. Describe what types of positive integers n can be written in the form $n = a^2 + b^2$ for integer a, b. For example, 450 can be written this way: $450 = 21^2 + 3^2$.

23. The number 450 has eight odd factors: 1, 3, 5, 9, 15, 25, 45, 75, 225. Odd factors can be split into factors in the form $4k + 1$ and $4k + 3$. 450 has 6 odd $4k + 1$ factors and 2 odd $4k + 3$ factors.
Well, it turns out there's a remarkable connection between the number of these types of factors and the *number of different ways* 450 (or any positive integer) can be written in the form $n = a^2 + b^2$ for integer a, b. 450 can be written many different ways ($a = 3, b = 21$ is one of them).
So, your task is to figure out what the rule is—then prove that it works (harder). Don't forget that a or b can be negative or zero.

24. Matsuura triangles, anyone?

8 *PCMI Squares*

Sketch of the Day

For each of these complex numbers, plot the number and its square in the same complex plane. You might want to draw segments from the origin to each number. If you need to measure lengths or angles, consider using Sketchpad.

$$\begin{aligned} J &= 3+i \\ U &= 1+2i \\ D &= \tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i \\ I &= i \end{aligned}$$

Keep doing examples until you can describe where the square is located in relation to the original number.

I'll take Bill Phill to block, please. Circle gets the square? Anyone?

Extension questions: what about cubes? square roots? reciprocals?

Important Stuff.

1. Let $z = \frac{12}{13} + \frac{5}{13}i$. Find and plot all of these on the same complex plane. (Estimate to three decimal places if you like.)
(a) z^0
(b) z^1
(c) z^2
(d) z^3
(e) z^4
(f) z^5

You mgiht not be experiencing deja vu here. Oh well.

2. Use Sketchpad to find the *angle* that forms when you go from one power of $z = \frac{12}{13} + \frac{5}{13}i$ to the next.

You mean it's the same angle each time? Uh, maybe. Or is it!

3. As the powers of $z = \frac{12}{13} + \frac{5}{13}i$ grow, eventually the powers work around the four quadrants, then back around into Quadrant I. Find the first power of z that re-enters Quadrant I.

4. Eileen takes powers of $z = \frac{12}{26} + \frac{5}{26}$ instead. What happens? Does anything stay the same?

5. If the complex number w is multiplied by a real number c (also called a *scalar*), what happens to the magnitude? the direction? What if c is negative?

6. If the complex number w is multiplied by i (also called i), what happens to the magnitude? the direction?

7. Let $w = 1 + i$ and $z = \sqrt{3} + i$. Find the magnitude and direction of wz, and compare the results to the magnitude and direction of w and z.

8. What, in general, is happening to magnitude and direction when you multiply two complex numbers?

We interrupt this problem set to hear from Wolfgang Bolyai, who sent this message to his son Janos:

> For God's sake, please give it up. Fear it no less than the sensual passion, because it, too, may take up all your time and deprive you of your health, peace of mind and happiness in life.

Apparently he didn't much care for Janos's interest in hyperbolic geometry.

We think Wolfgang was a real estate agent.

Neat Stuff.

9. Can a regular octagon ever have eight 90-degree angles? Mel wants you to explain.

10. Was there a time in Monte's life when his weight in pounds was exactly equal to his height in inches? (We don't mean rounding off to the nearest inch or pound or anything, either.)

Unless Monte was an incredibly small baby. We don't think so.

11. Let a be a complex number with magnitude 5, and b be a complex number with magnitude 13. Consider $a + b$: what could its magnitude be? Do you add magnitudes when you add complex numbers, or what?

12. Suppose m can be written as the sum of two squares, and n can also be written as the sum of two squares. Prove that mn can *also* be written as the sum of two squares.

13. Consider the equation $y = x^2 - 12x + 41$.
(a) Sketch the graph of the equation.
(b) Explain how you know (from the graph) that there are no real solutions to the equation $x^2 - 12x + 41 = 0$.

14. The graph of $y = x^2 - 12x + 41$ is a parabola. Consider the parabola with the same vertex that opens in the opposite direction.
(a) Find the equation for this parabola.
(b) This parabola intersects the x-axis at what two points?
(c) The equation $x^2 - 12x + 41 = 0$ has two solutions in complex numbers. Show that these solutions are related to the x-intercepts of this reflected parabola.

15. Plot the solutions of each equation in the complex plane.
(a) $x^2 - 12x + 32 = 0$
(b) $x^2 - 12x + 35 = 0$
(c) $x^2 - 12x + 36 = 0$
(d) $x^2 - 12x + 37 = 0$
(e) $x^2 - 12x + 40 = 0$
(f) $x^2 - 12x + 85 = 0$

As Tattoo might say... oh, it's too easy, never mind.

16. Suppose a regular n-gon is placed with its center at the origin and one of its vertices at $(1, 0)$.
(a) For what values of n will $(-1, 0)$ be a vertex of the n-gon?
(b) For what values of n will $(0, -1)$ be a vertex of the n-gon?
(c) Find all points that are on the regular 12-gon but aren't on any smaller n-gon.

17. Take all the points on the regular 12-gon and square them. What happens?

Topological Stuff.

18. For each of these objects, count the number of **vertices**, **edges**, and **faces** of the objects.
(a) a cube
(b) a tetrahedron

FREE SLURPEE DAY! FREE SLURPEE DAY!

(c) a cylinder
(d) a square pyramid
(e) a dodecahedron
(f) a football
(g) a shape provided by Joyce (bonus points available by degree of difficulty)

Not that football, PCMI is an international conference. So who you got, France or Italy?

19. Make a table with the number of vertices, edges, and faces for these objects, or any others you enjoy. (A sphere, perhaps?) Is there a meaningful relationship between these numbers that holds no matter what solid you start with?

20. What about a graph in the plane, like Day 1's sketch, or even just a parallelogram? Is there some pattern to the vertices, edges, and faces there?

Here a **face** would just be an enclosed area in the plane.

Tough Stuff.

21. Hey, it's another division problem where we've provided one of the digits, an 8. The other digits are unknown. No leading digit is a zero, we swear we really mean it this time.

```
                x x x x x
        _____________________
 x x x | x x x x x x x x
         x x x
         -----
                x x x x
                  x x x
                  -----
                  x x x x
                  8 x x x
                  -------
```

22. $\frac{J}{MI} + \frac{O}{GU} + \frac{Y}{EL} = 1$. Here, each letter is a unique and distinct number from 1 to 9. Find a solution!

23. As you keep taking powers of $z = \frac{3}{5} + \frac{4}{5}i$, they keep wrapping around in a circle. But will they eventually wrap around onto themselves? In other words, are there powers k and m with $z^k = z^m$ for this z?

24. Steve from Cincinnati promised Matsuura triangles, then took it back! Who's got 'em!

25. Sketch the graph of the equation $x^y = y^x$ in the first quadrant.

65 *(to the) 1/2: Snakes on a Hyperbolic Plane*

Sketch of the Day

Use the narrowest part of a ruler to construct a triangle made up of "straight" segments on the hyperbolic paper. How does the triangle compare to one in the Euclidean plane?

Use a compass to make a circle on the hyperbolic paper.

Ben is hyper for the hyperbolic plane. Woo.

Spread out that paper! There's plenty of real estate to go around.

Important Stuff.

1. Working with a group (or perhaps a subgroup with more than 1 element), discuss how the definitions of each of these terms might change in the hyperbolic plane. Be sure to agree upon the definitions on the Euclidean plane before talking about definitions in the hyperbolic plane.
(a) parallel
(b) perpendicular
(c) circle
(d) equilateral triangle
(e) midpoint
(f) congruent triangles

Cathy and Kathleen could form such a subgroup, perhaps along with the similarly-named participant from Cincinnati?

2. In the first week, we constructed an equilateral triangle in the Euclidean plane using a straight edge and compass (effectively, using the Sketchpad tools). Try to perform the same construction in the hyperbolic plane. Does it work? Does it produce an equilateral triangle? Compare the triangles you and others at your table made: are they

Is $65^{1/2}$ close to route 66?

congruent? similar?

3. In the Euclidean plane, two lines that are perpendicular to the same line are parallel. Is this still true in the hyperbolic plane?

4. If line MI is parallel to line CH, and that line is parallel to line EL, is line LE parallel to line MI? Oh, this is in the hyperbolic plane.

If the world were hyperbolic, would McAllen and Cincinnati be closer or further away?

Neat Stuff.

5. Is it possible for two triangles to be similar but not congruent?

6. What's a parallelogram?

7. Is the Midline Theorem true? This is the theorem that says that if you take two midpoints in a triangle and connect them, it's parallel and half as long as the third side.

8. What happens to the Pythagorean Theorem?

9. How might you go about constructing a regular octagon with 90° angles?

10. Construct a regular hexagon. Extra credit: construct a regular hexagon with six right angles.

Tough Stuff.

11. Use the hyperbolic paper to solve and prove the equivalent problem to Art's tent problem on Day 2.

PROBLEM

Isn't the hyperbolic plane wacky?

9 *A Tale of Cincinnati and McAllen*

Sketch of the Day

Use the provided Sketchpad model (and tools) to construct an equilateral triangle. Find its side lengths and angle measures *using the provided hyperbolic tools and not the Measure menu.*

What happens when you make the circle larger? smaller?

Careful, don't make the circle smaller than the points you've defined. All heck breaks loose. Oh, now you're going to do it for sure.

Important Stuff.

1. Suppose complex numbers z and w are plotted. Describe how you could use the picture to figure out where to plot the sum $z + w$.

2. (a) Armando's favorite complex number is $4 + 3i$. Find the magnitude and direction of the *square* of this number.

(b) Armando's favorite complex number is $\frac{4}{25} - \frac{3}{25}i$. Find the magnitude and direction of the *square* of this number.

It's okay if the direction isn't between 0 and 360°.

(c) The two Armandos meet, and their numbers are multiplied together. What happens?

3. Here's three complex numbers: $m = 3+i, e = 3+2i, i = i$.

(a) Plot these three numbers in the complex plane.

(b) Multiply each number by $1+2i$, then plot the results in the complex plane.

(c) Describe how to relate the new numbers to the old.

4. (a) Find the magnitude and direction of $w = 2 + i$ and $z = 3 + i$.

(b) Find the magnitude and direction of the product wz.
(c) How can the magnitude and direction of wz be determined by the information about w and z?

5. Show that this is true:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Sendhil says it must be true, since it's in the problem set. Bah.

Neat Stuff.

6. Use the result in problem 5 to prove that when two complex numbers are multiplied, their magnitudes are multiplied.

7. Suppose x and y are complex numbers where $xy = 10i$. Find some numbers $x = j + ui$ and $y = a + ni$ where none of j, u, a, n are zero. Find some more? A general rule?

8. Suppose z is a complex number with magnitude 1 and direction θ. Then $z = a + bi$ with $a = \cos\theta$ and $b = \sin\theta$.
(a) Calculate z^2 directly by squaring $z = a + bi$. Notice anything?
(b) Find a formula for $\cos 3\theta$.

If you're not working with trigonometry regularly, these two will be extremely boring problems.

9. Suppose z and w have magnitude 1, z has direction α and w has direction β. Let $z = a + bi$ and $w = c + di$.
(a) Calculate zw.
(b) What can be said about the magnitude and direction of zw?
(c) What's the formula for $\sin(\alpha + \beta)$?

10. Use Sketchpad to construct a hyperbolic triangle. Then, find the one point that minimizes the sum of the distances to the three vertices of the triangle. Remember to measure distance using the provided tools and not the Measure menu. What's so special about this point, anyway?

Tough Stuff.

11. Construct a regular octagon in the hyperbolic plane.

12. Construct two different triangles in the hyperbolic plane *with the same area*, or prove that such a construction is impossible.

FREE SLURPEE DAY! FREE SLURPEE DAY!

10 *Once Upon a Time in PCMI*

Sketch of the Day

Plot the complex number $z = \frac{3}{5} + \frac{4}{5}i$ in the plane. Then, use Sketchpad to plot the powers of z: z^2, z^3, until they wrap around back into Quadrant I.

Repeat the sketch starting with $z = \frac{2}{3} + \frac{2}{3}i$. What changes? Why?

Important Stuff.

1. Use Sketchpad to show that the four vertices 0, z, w, and $z + w$ form a parallelogram in the complex plane.

2. Let $z = 2 + i$. Plot each of the following on the same complex plane.
(a) z
(b) $2z$
(c) $3z$
(d) $-z$
(e) kz where k is any integer

Change the scale in Sketchpad by dragging the unit point $(1, 0)$. Other points should scale so they keep the same coordinates. For $-z$, what operation might be best?

3. What happens to the magnitude and direction of a complex number when Dave multiplies it by i? You may assume that Dave carries out the multiplication correctly.

4. Let v be a complex number with magnitude 2.

(a) Draw a shape to indicate where v could lie in the complex plane.

(b) Pick a value of v and square it: where does it go? Describe all the possible places where v^2 could lie.

Here are some values of v with magnitude 2: $\sqrt{3}+i$, $\frac{6}{5}+\frac{8}{5}i$, $\sqrt{2}+\sqrt{2}i$, $\frac{24}{13}+\frac{10}{13}i$. You want more, ask Henri. Don't forget that Sketchpad allows you to construct a point "on" an object.

5. Find the magnitude and direction of $z = 2 + i$, of $w = 3 + 2i$, of zw. (Find exact values for the magnitude, and approximate the direction to two or three decimal places.)

6. Guess the magnitude and direction of each of these, then check a few of them. Remember, $z = 2 + i$ and $w = 3 + 2i$.

(a) z^2

(b) zw^2

(c) zw^3

(d) zw^{-1}

(e) $(z^2w^3)^0$

Neat Stuff.

7. Show that this is true:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Brian says it must be true, since it was also on Day 9.

8. Suppose m can be written as the sum of two squares, and n can also be written as the sum of two squares. Prove that mn can *also* be written as the sum of two squares.

9. Use the result in problem 7 to prove that when two complex numbers are multiplied, their magnitudes are multiplied.

10. If $z^2 = 11 + 60i$, what is z? Is it possible for Matt and Matt to get different correct answers to this question? In general, given a complex number's magnitude and direction, find the magnitude and direction of all possible square roots.

11. If $z^3 = -117 + 44i$, what is z? Is it possible for Greg, Carole, and Peter to all get different correct answers to this question? In general, given a complex number's magnitude and direction, find the magnitude and direction of all possible cube roots.

12. Suppose the sum and product of two complex numbers z and w are both real. Prove that either z and w are both real numbers, or $w = \overline{z}$.

13. Use multiplication of complex numbers to find a formula for $\tan(\alpha + \beta)$ in terms of $\tan \alpha$ and $\tan \beta$.

14. Find two values of $\tan \alpha$ and $\tan \beta$ that make $\tan(\alpha+\beta) = 5$.

15. Calculate this:

$$\frac{(5+i)^4}{(239+i)}$$

What the heck could this possibly be useful for?

16. **Hyperbolic Sketch of the Day!** Use the Poincare sketch to construct an equilateral triangle, then pick a point inside. Construct the three altitudes from that point to the three sides of the triangle. Notice anything?

17. Craig looks at a sequence of geometric shapes:

point, line segment, square, cube, hypercube (!)

These shapes have parts that have dimension 0 (points), dimension 1 (edges / segments), dimension 2 (faces), dimension 3 (solids), dimension 4 (hypersolids?). How many of each for each shape? Patterns, anyone? Dimension 5? Dimension n?

Tough Stuff.

18. Let S be the set of complex numbers z with magnitude 2. Find a function (such as $z \mapsto z^2$) that produces an ellipse (and *not* a circle) as the output when S is used as the input.

19. Complex number w is a member of set M if the rule $z \mapsto z^2 + w$ with starting point 0 never has $|z| > 2$.

(a) Find some numbers w that are in set M, and some that aren't.

(b) Prove that if $|w| > 2$, it cannot possibly be in set M.

(c) What does set M look like if it is graphed in the complex plane?

20. What shapes tesselate in the hyperbolic plane?

21. (a) Evaluate $\tan 89°$ to two decimal places.

(b) How many degrees are in 1 radian? Give your answer to three decimal places.

(c) What is going on here? Is this a coincidence?

Coincidence? I think not!

11 *The Devil Went Down To PCMI*

Sketch of the Day

Let z be a complex number with magnitude 2. Draw a shape to indicate where z could lie in the complex plane, then describe and plot the path z^3 takes through the complex plane.

Move z around to the three different places that are solutions to the equation $z^3 = 8$. Where are they located in relation to one another?

Today's title could also have been "The Devil Wears PCMI" but we decided against that.

Today's correct use of the word "different" is brought to you by Neat Stu's Important Stuff.

Important Stuff.

1. (a) What do the powers of $z = \frac{3}{4} + \frac{3}{4}i$ look like in the complex plane?
(b) What do the powers of $w = \frac{2}{3} - \frac{2}{3}i$ look like in the complex plane? Why does it go that way instead of that way?

Richard says it looks like a buncha points. Not a very exciting answer.

2. (a) Find the three solutions to $x^3 - 1 = 0$. (Factor?)
(b) Find the three solutions to $x^3 = 64$.

3. Without Sketchpad, graph the line $y = 4x + 2$. On the same axes, graph the output of each function when that *line* is used as input. A table of values might help.
(a) $(x, y) \mapsto (2x, y)$
(b) $(x, y) \mapsto (x, \frac{y}{2})$
(c) $(x, y) \mapsto (x - y, x + y)$

So this is a function that takes a *line* and outputs another... line? shape? I guess it depends!

4. Let S be the circle of all the complex numbers with magnitude 2. (Plot one of them and call it z.) Nagwa and others have seen what the function $z \mapsto z^2$ will look like when you use this circle as input. So what about a different function:

$$z \mapsto z^2 + z$$

What does the output look like when S is the input?

5. Describe how you could use the picture in problem 4 to explain that $z = 2$ is a solution to the equation

$$z^2 + z = 6$$

Neat Stuff.

6. Let $z = 34 + i$ and $w = 55 + i$. The product zw can be written as a scalar multiple of another complex number in the form (something) $+ i$. What's the something? (Oh, what about $\frac{z}{w}$?)

34 and 55, you say? Never seen *those* numbers before. Perhaps the answer is 89.

7. Let $a = 2 + i$, $v = 5 + i$, $r = 13 + i$, $y = 21 + i$. Find the direction of the product $avry$. Can this be generalized?

8. (a) Expand the expression $(x + 2)^3$.
(b) How many faces are there on a cube? How many edges? How many vertices? Wacky.
(c) Does this pattern continue at all, in lower or higher dimensions?

9. Jeff challenges you to find the three solutions to $x^3 = i$. What do you say to this!

10. (a) Multiply out $(a + bi)^3$.
(b) If z is a complex number with magnitude 1 and direction θ, what are its coordinates? (If you're not a trig fan, just move along, nothing to see here.)
(c) Write rules for $\cos 3\theta$ and $\sin 3\theta$ based on the first two parts of this problem.
(d) The rule for $\cos 3\theta$ is $4\cos^3\theta - 3\cos\theta$. ¿Que pasa?

11. Take a number z in the complex plane. How can you construct the conjugate $\overline{z}$?

Take a number? Is this a deli?

12. Let S be the set of complex numbers z with magnitude 2. Find a function that produces an ellipse (and *not* a circle) as the output when S is used as the input.

13. **Hyperbolic Sketch of the Day!** Use the Poincare sketch to investigate Art's tent problem on Day 2. Is the solution the same? Is the *construction* the same?

14. **Sketchy Sketch of the Day!** Debbie did some division in the complex plane yesterday. Take $w = 6 + 8i$ and start dividing it *into* stuff: lines? segments? triangles? circles? Use a "Locus" construction in Sketchpad to see what the results look like. When you divide w by a triangle, what does the new shape remind you of?

This is today's Rorschach test of the day, I guess.

15. Suppose $\tan A = \frac{1}{x}$ and $\tan B = \frac{1}{y}$, where x and y are integers. Is it possible for $\tan(A + B)$ to also be in the form $\frac{1}{z}$ for some other integer z? If so, where?

16. Let z be a complex number with magnitude 1. If $z = x+yi$ and $y > 0$, explain how you know that $y = \sqrt{1 - x^2}$.

17. Let $z = x + (\sqrt{1 - x^2})i$ as in the last problem. So you think you've seen it all, eh?

(a) Expand z^2 and look at the real part. Hm? Zeros?

(b) Expand z^3 and look at the real part. Hmmm? Zeros? Connection to some earlier question, perhaps?

(c) Expand z^4 and look at the real part. Hmmmmm?

(d) Graph the real part of z^2, z^3, z^4 as a function of x.

(e) What might z^n look like? Okay, this really should've been Tough Stuff, but too bad.

The TI-89 can handle this, believe it or not. Good ole 89.

Pafnuty says hi. Who's that? Look it up!

Tough Stuff.

18. Find some connections between what we've been doing with tangent and Taylor series.

19. Ask Peg about the Hyperbolic Law of Cosines.

20. Use the Taylor series for $\sin x$, $\cos x$, and e^x to show that $e^{i\pi} + 1 = 0$. Woo!

21. What is the value of i^i? (Use a TI-89 in radian mode.) How on earth would one arrive at such a thing?

22. Ooh, calculus: Let $f_n(x) = \cos x \cos 2x \ldots \cos nx$. For which n in the range $1, 2, \ldots, 10$ is the integral from 0 to 2π of $f_n(x)$ non-zero?

23. Solve the problem available online at http://www.claymath.org/millennium/Riemann_Hypothesis/ then claim your prize.

Yes, yesterday was actually Free Slurpee Day. We hope you believed us.

12 *My Own Private PCMI*

Important Stuff.

What? The Important Stuff is first? Guess you better do that first, then!

1. Let z be a complex number with magnitude 3. Plot a circle that contains z, allowing for the radius to be changed. If set S is this circle, find the output when S is the input to the rule

$$z \mapsto z^2 - z$$

2. *Use the sketch from the last problem* to find one solution to each equation.
(a) $z^2 - z = 6$
(b) $z^2 - z = 12$
(c) $z^2 - z = -9$ (approximate quick)
(d) $z^2 - z = 20$

As Roy Scheider might say, "We're going to need a bigger circle."

3. Let S be a circle with magnitude 2. What's the output look like when S is the input to the rule

$$z \mapsto z^3 + z + 2$$

4. Adapt the sketch in the last problem to find or approximate the *three different* solutions to the equation

$$z^3 + z + 2 = 0$$

5. Consider the rule

$$z \mapsto z^2 + 4z + 5$$

(a) Susan uses a really small circle S as input. What does the output look like? By small we mean small.
(b) Ralph uses a really large circle S as input, and grabs Susan's circle, dragging it until it's huge. What happens? Does the output ever cross the origin? How many times?

Yes, yesterday was actually Free Slurpee Day. We hope you believed us.

Sketch of the Day

Go back and do the Important Stuff. Oh you're done? Fine then, find (approximately) the five complex numbers that make $z^5 = 1$. Don't do this by factoring, it's the Sketch of the Day! While you're at it, find the five complex numbers that make $z^5 = -1$. That's all.

Not So Critically Important Stuff.

6. (a) Plot the two solutions to $x^2+6x = -25$, using Sketchpad or otherwise.
(b) Plot the two solutions to $4x^2 + 12x = -25$.
(c) Plot the two solutions to $9x^2 + 18x = -25$.
(d) What's going on, is there a pattern here?

7. Prove that any quadratic equation with real coefficients must have exactly two roots in the complex plane, except when those roots are the same point.

8. Consider the function $f(x) = x^3 + 6x + 3$. Don't worry, we're not going to ask you anything about the complex plane in this problem.
(a) Find a number a so that $f(a) > 0$.
(b) Find a number b so that $f(b) < 0$.
(c) Explain how you know that there *must* be a real number c so that $f(c) = 0$.

9. Suppose a function $g(x)$ has $g(a) > 0$ and $g(b) < 0$. Must there always be a c so that $g(c) = 0$? Explain or provide examples.

Neat Stuff.

10. Do some division! Build a triangle in the complex plane, then a point outside the triangle. Using locus constructions, divide the point into the triangle. You're going to need 3 locuses, loci, loca, whatever, one for each segment of the triangle. What does the resulting shape look like?

So to do the division, click on the isolated point first, then the point on the edge of the triangle.

11. What changes when the origin is inside the triangle? Why?

12. What changes when the origin is *on* the triangle? Why?

Yes, yesterday was actually Free Slurpee Day. We hope you believed us.

13. What about angles in that last construction? How would angles even get measured in these absurd new shapes?

There should be plenty of Neat and/or Tough Stuff to go around from Set 11, too. Feel free to try any group of these problems. Although many are related to the core goals of the course, most are side quests to try whenever. Enjoy!

14. The function $f(z) = z^3 + 4z$ is an *odd* function. What would an odd function look like under these complex mappings we've been doing?

15. While you're at it, what about even functions?

16. Use the results from the Sketch of the Day to find the ten numbers that make $z^{10} = 1$.

17. Let $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. The domain of $f(x)$ is real numbers with $-1 \leq x \leq 1$.

Look out, it's a Taylor series! RUN! Oh, sorry, it's only a Maclaurin series. False alarm.

(a) Approximate $f(\frac{1}{5})$ to six decimal places. You might need to take a few terms.

(b) Approximate $f(\frac{1}{239})$ to six decimal places. More terms, or less terms?

(c) Evaluate $16f(\frac{1}{5}) - 4f(\frac{1}{239})$ to five decimal places.

(d) Hey, we forgot: find the direction of the complex number

$$\frac{(5+i)^4}{239+i}$$

Wacky.

18. Consider this set of polynomials: $P_0(x) = 1$, $P_1(x) = x$ to start. From then on, polynomials are generated by the rule

$$P_n(x) = 2x \cdot P_{n-1}(x) - P_{n-2}(x)$$

Investigate these polynomials: what is their behavior? What roots do they have?

19. Last night's Pizza & Problem Solving offered a question about numbers like 101, 10101, 1010101, etc. Try squaring these numbers and see what happens. Can you prove it?

Tough Stuff.

20. Turns out that $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$. That's not tough. The tough part is to factor that quartic piece into two quadratic factors, where each quadratic factor might have real coefficients (not just integers or fractions). Does this give you any insight into how to *construct* a regular pentagon?

Is that the Fundamental Theorem of Algebra? Man oh man.

13 *Sweet Home PCMI!*

Sketch of the Day

Investigate the behavior of the function $f(z) = z^2 - 2z + 4$ as it operates on different "magnitude circles" (circles centered at 0). Estimate with very poor accuracy the two solutions to $z^2 - 2z + 4 = 0$. (They are both on the same magnitude circle.)

Investigate the same behavior for the function $f(z) = z^2 - 4z + 4$.

Today's problem set is inspired by Lynyrd, not Reese, and is brought to you by the letter i, and the number i.

Not to worry, the plague of locus is almost complete.

Important Stuff.

1. What two numbers have a sum of 2 and a product of 4?

2. Consider the function $f(x) = x^3 + 6x + 3$. Estimate $f(\frac{1}{100})$ and $f(100)$.

3. Consider the function $f(z) = z^3 + 6z + 3$.

(a) What does the output of $f(z)$ look like when you use a very small magnitude circle?

(b) What does the output of $f(z)$ look like when you use a very large magnitude circle?

(c) Explain how you know that there *must* be a magnitude circle that contains a point z with $f(z) = 0$.

Say, this is the same function we used in the last problem!
If you can't tell, your circle isn't small enough yet.

4. Let $f(x) = x^3 + 12x - 4$. Give a good, quick estimate for each of these. Emphasis on "quick".

(a) $f(\frac{1}{100})$
(b) $f(-\frac{1}{100})$
(c) $f(100)$
(d) $f(-100)$

Is that the Fundamental Theorem of Algebra? Man oh man.

5. Shenaz considers the function $f(z) = z^3 + 12z - 4$. *Without using Sketchpad*, she and the rest of you shall answer the questions of problem 3 about this other function.

6. Convince yourself, Rebecca, and everyone else that this is true:

> Let $f(z)$ be a polynomial of degree n. Then there must be at least one solution to the equation $f(z) = 0$ in the complex plane.

"Degree n" just means that the polynomial starts $f(z) = az^n + \cdots$

Neat Stuff.

7. Consider the function $f(z) = z^2 - z$.

(a) Find a solution to $z^2 - z = 1$ using Sketchpad.

(b) Find a way to use this sketch to build a golden rectangle. Is there more than one length in this diagram that could produce a golden rectangle?

8. Use Sketchpad to approximate the five solutions to $x^5 + 2x^2 + 10 = 0$. How many of the roots are real numbers?

9. Use Sketchpad to approximate the four solutions to $x^4 + 3x^2 + 1 = 0$. Yes, there are four solutions and not two. What happens, in general, to even functions? (Tougher: Find the roots by factoring.)

10. Track down the solutions to $x^3 - 7x^2 + 15x - 9 = 0$. What does a "double root" look like in the complex plane? How many solutions does this equation have?

11. Once you calculate the roots of a given polynomial, try plotting them as points in the complex plane, then seeing how the function behaves as it wanders by them. Or you could try working with circles that aren't centered at the origin. Check out

What's a "winding number"?

http://www.keypress.com/sketchpad/general_resources/recent_talks/complex_ictmt6/downloads/complex_functions.pdf

12. **Continuity!** On Saturday Donald woke up at 8 am, climbed to the top of Mt. Timpanogos, then camped out there overnight. On Sunday, Donald woke up at 8 am, then climbed down the same trail he climbed up. He didn't climb at any consistent speed, and stopped to take in the views many times. Was there definitely some time during

Is that the Fundamental Theorem of Algebra? Man oh man.

the day that Donald was in the same spot (on the trail) at the same time on both Saturday and Sunday?

13. The current record for calculating π is into the trillions of digits. One completely wacky formula for calculating π comes from this calculation:

$$\frac{(N+i)^{12} \cdot (57+i)^{32} \cdot (110443+i)^{12}}{(239+i)^5}$$

Oh wait, we forgot N. Find the integer N so that this entire product has direction exactly 45°. This formula was discovered by a high school teacher in 1982. Perhaps you could find some more?

Review Your Stuff.

Historically the final day is considered review. Because this is a self-reflective process of discovery, we think that an end product of this discovery might be some summarizing questions of what you might find valuable in this course. We would like these to get at what you think are important mathematical themes in our course, and also themes that might apply to what you teach in your own school. We hope this will be a valuable journey, but mostly we're just lazy and want your (*table*) to

Is there really such a thing as a "self-reflective process of discovery"? (Google says no.)

- write two problems on any subject that has cropped up this year, and
- write a potential title for Day 14's problem set.

Hey, creativity counts! We'll include many of the best problems on Day 14's set, and the best title. Feel free to poke fun at easy targets like Art or Steve.

Your table's problems will be judged on a sliding scale that includes "Important", "Neat", "Wacky", and "Useless", among others.

Tough Stuff.

14. Break $x^5 - 1$ into quadratic factors, then use the factorization to *construct* a regular pentagon with straight edge and compass (or, use the Sketchpad tools).

15. Investigate the behavior of the function $z \mapsto z + \frac{1}{z}$ in the complex plane. An interesting paper on this and related topics can be found online at:

http://jwilson.coe.uga.edu/olive/Joukowski.Web/Joukowski.Paper.html

This paper includes some references to previous works by some guy from Washington.

14 *Escape From PCMI*

Or, "The PCMI Redemption." Or, "Raiders of the Lost Arctan." Or, "2006: A PCMI Odyssey." Or, "i, Robot." Or, "A PCMI Too Far." Or. . .

Our Stuff.

0. Is there any reason that a polynomial of degree n can't have *more* than n roots?

0. So, a cubic equation has at least one solution. Explain how factoring could help you show that there are a total of three solutions to the cubic equation.

Here a "repeated root" might be counted twice or more.

Polynomial Stuff.

1. Find all solutions to each quadratic equation.
(a) $w^2 - 26w + 169 = 0$
(b) $w^2 - 26w + 168 = 0$
(c) $w^2 - 26w + 167 = 0$
(d) $w^2 - 26w - 1 = 0$
(e) $w^2 - 26w + 170 = 0$

8, 11. Let $f(x) = -8x^4 - 9x^3 + 4x^2 + 3x + 8$ and $g(x) = 9x^3 - x^2 - 6x - 2$. Find the remainder when each of these is divided by $x^2 + 1$.
(a) $f(x)$
(b) $g(x)$
(c) $f(x) + g(x)$
(d) $f(x) - g(x)$
(e) $g(x) + 19$
(f) $f(x) \cdot g(x)$

9. (a) Construct a quadratic equation with roots $2 + i$ and $2 - i$. (Sum and product?)

No mathematicians were harmed during the making of this course.

(b) Harder: construct a quadratic equation with roots $2+i$ and $-2+i$.

7. The graph of $y = x^2 + 10x + 27$ is a parabola. Look back at problem 14 from Day 8 and answer the same questions about this graph.

Geometric Stuff.

11. Use Sketchpad to build a triangle OIL. Then plot each of these:

- the *orthocenter*, intersection of the altitudes
- the *centroid*, intersection of the medians
- the *incenter*, intersection of the angle bisectors
- the *circumcenter*, intersection of the perpendicular bisectors

Which one is the odd man out?

Since all three of each type intersect in the same point, you only need to plot two, then find the intersection. Whoomp, there it is.

8. Triangle $A(2,3), R(4,7), T(-5,1)$ needs to transform into triangle $P(8,18), E(18,40), G(-3,-11)$.

(a) Write a transformation in the form $(x,y) \mapsto (ax+by, cx+dy)$ that does this.

(b) Is there only one such transformation?

2.
- Cut a strip of paper about 1.5 cm wide.
- Tie a knot near the end of the strip.
- Flatten everything to form a polygon. Where have you seen this shape? (Powers of z?)
- Fold the strip around the polygon.
- Tuck the end in.
- Puff up the shape by pressing the midpoint of each side using a fingernail.

Our hope is that Table 2 will prepare an example of this construction.

1. Slider Man, governor of Hyperbolia, wants to build thhe WTC (Wormhole Transit Center) somewhere between the three cities of Hyperion, Balloonium, and Averium. (Triangle BAH?)

Given the space is hyperbolic, where should the WTC be built so as to minimize the total length of the three new geodesic roads?

This problem seems familiar somehow, I just can't place it...

6. Does the Pythagorean Theorem carry over into the hyperbolic plane? What about the Side-Angle Inequality (longest side opposite longest angle)?

No mathematicians were harmed during the making of this course.

Complex Stuff.

8. Describe in detail how the locus sketch is used to solve equations in the complex plane. Include:
(a) What's the meaning of the green circle?
(b) What's the meaning of the red locus?
(c) Which is first, magnitude or direction of solutions?
(d) Why wouldn't we do that in reverse order?

5. (a) Find the 12 solutions to $x^{12} - 1 = 0$.
(b) If one vertex of a regular dodecagon is $(1, 0)$ and the center of the dodecagon is $(0, 0)$, what are the others?

9. How many complex numbers $a + bi$ have integer a, b and magnitude $\sqrt{325} = 5\sqrt{13}$? What of $\sqrt{1105}$?

10. Approximate the three solutions to the equation

$$x^3 + ix^2 - 2x = i$$

3. Solve $z^5 + 2z^4 - 3z^3 + 2 = 0$ in any way you can. (Must do it graphically.)

4. (a) Take a complex number z, then continue squaring it. What could happen, depending on what z is?
(b) (Tougher.) Start with a complex number c. Square it, then add c. Then square that, and add c. Keep doing that. What could happen, depending on c?

Wacky Stuff.

0. What number is this?

$$1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}$$

0. What number is this?

$$2 + \cfrac{1}{1 + \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\ddots}}}}}$$

Here the numbers on the outside go $2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots$.

7. Amanda's favorite function is $A(x) = x^4 - 3x^2 + x^2 + 2$. Find a cubic function, a quadratic function, and a linear function that are all tangent to $A(x)$ at $x = 1$.

No mathematicians were harmed during the making of this course.

10. (a) Find a third or fourth-degree polynomial tangent to $y = x^5 - x^3 + 3x - 7$ at $x = -1$ *and* $x = 2$.

(b) Find the one fourth-degree polynomial that is tangent at $x = -1$ and $x = 2$, and also passes through the point $(0, -7)$.

11. Consider the following:

$$7^2 + 21^2, 9^2 + 27^2, 13^2 + 39^2, \ldots, n^2 + (3n)^2$$

$$7^2 + 14^2, 9^2 + 18^2, 13^2 + 26^2, \ldots, n^2 + (2n)^2$$

Do you notice any patterns? Can these patterns be used to generate Pythagorean triples?

11. Check these out:

$$45^2 + 46^2 = ?, 37^2 + 67^2 = ?, 53^2 + 25^2 = ?$$

Is there a pattern? Can you generalize or prove it?

5. How many Reidemeister moves does it take to untangle $f(z) = z^5 + z^4 + z^3 + z^2 + z + 1$ where z is a complex number on the function $f(x) = \cos x$? Assume that the ends of $f(z)$ are connected by a straight line.

What's a Reidemeister move, you say? And how do we know which parts of the path are over or under other parts? Uhhh.

8. Extend the sketches from the third week to allow for magnitude circles that aren't centered at the origin. Try this with $f(z) = z^3 + 1$ and see what happens. (Where might be a good place to center the circle, then?)

Tough Stuff.

12. Build a polygon in the complex plane, then a point outside the polygon. Using locus constructions, divide the point into the polygon. What could the resulting shape look like for different polygons?

You will need several loci, one for each segment. Or try merging? Ben might be lying about this.

5. Consider the dodecagon from problem 14. If each vertex of the dodecagon is transformed by the rule

$$(x, y) \mapsto (3x - 2y, 4x + 5y)$$

find the exact area of the new shape formed.

6. Assuming that Matsuura triangles exist, prove that one side length of the triangle must be a multiple of 24.

Thanks, everyone! It's been a real treat teaching the class and meeting you all. See you again soon.